CONCENTRATION OF THE INTEGRAL NORM OF IDEMPOTENTS

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Abstract. This is a companion paper of a recent one, entitled *Integral concentration of idempotent trigonometric polynomials with gaps*. New results of the present work concern L^1 concentration, while the above mentioned paper deals with L^p-concentration.

Our aim here is two-fold. At the first place we try to explain methods and results, and give further straightforward corollaries. On the other hand, we push forward the methods to obtain a better constant for the possible concentration (in L^1 norm) of an idempotent on an arbitrary symmetric measurable set of positive measure. We prove a rather high level $\gamma_1 > 0.96$, which contradicts strongly the conjecture of Anderson et al. that there is no positive concentration in L^1 norm.

The same problem is considered on the group $\mathbb{Z}/q\mathbb{Z}$, with q say a prime number. There, the property of absolute integral concentration of idempotent polynomials fails, which is in a way a positive answer to the conjecture mentioned above. Our proof uses recent results of B. Green and S. Konyagin on the Littlewood Problem.

1. Introduction and statement of results

The problem of p-concentration on the torus for idempotent polynomials has been considered first in [1], [2], [4], [7]. We use the notation $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ for the torus. Then $e(t) := e^{2\pi it}$ is the usual exponential function adjusted to interval length 1, and we denote e_h the function $e(ht)$. For obvious reasons of being convolution idempotents, the set

$$\mathcal{P} := \left\{ \sum_{h\in H} e_h \ : \ H \subset \mathbb{N},\ \sharp H < \infty \right\} \tag{1}$$

is called the set of *(convolution-)idempotent exponential (or trigonometric) polynomials*, or just *idempotents* for short. The p-concentration problem comes from the following definition.

Definition 1. *Let $p > 0$. We say that there is p-concentration if there exists a constant $\gamma > 0$ so that for any symmetric (with respect to 0)*

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measurable set E of positive measure one can find an idempotent $f \in \mathcal{P}$ with

$$\int_E |f|^p \geq \gamma \int_{\mathbb{T}} |f|^p. \tag{2}$$

The supremum of all such constants γ will be denoted as γ_p, and called the level of p-concentration.

The main theorem of [3] can be stated as:

Theorem 2 (Anderson, Ash, Jones, Rider, Saffari). *There is p-concentration for all $p > 1$.*

We prove in our recent paper [5] that there is p-concentration for all $p > 1/2$, while the same authors conjectured that idempotent concentration fails already for $p = 1$. Moreover, we prove that the constant γ_p is equal to 1 when $p > 1$ and p is not an even integer. This is in line with the fact that L^p norms behave differently depending on whether p is an even integer or not in a certain number of problems, such as the Hardy-Littlewood majorant problem (*does an inequality on absolute values of Fourier coefficients imply an inequality on L^p norms?*) or the Wiener property for periodic positive definite functions (*does a positive definite function belong to L^p when it is the case on a small interval?*). The fact that one can find idempotents among counter-examples to the Hardy-Littlewood majorant problem had been conjectured by Montgomery [11] and was recently proved by Mockenhaupt and Schlag [10], and we rely on their construction in [5]. At the same time, we were able to revisit the Wiener property in order to construct counter-examples among idempotents [6].

Even if we disproved the conjecture of [3] for $p = 1$, the situation is not yet entirely clear. Indeed, the constant γ can be taken arbitrarily close to 1 when we restrict the class of symmetric measurable sets to symmetric open sets or enlarge the class of trigonometrical polynomials to all positive definite ones, that is, allow all non negative coefficients and not only 0 or 1. So one may conjecture that $\gamma_1 = 1$ (even if we understand that one should be cautious with such conjectures). By pushing forward our techniques, we improve our previous constant and prove the following.

Theorem 3. *For $p = 1$ there is concentration at the level $\gamma_1 > 0.96$. Moreover, for arbitrarily large given N the corresponding concentrating idempotent can be chosen with gaps at least N between consecutive frequencies.*

In order to prove this theorem, we will describe the main steps of our proofs in [5] before focusing on the improvements. When doing this, we also give a relatively simple proof of the fact that the best constant

γ_2 for symmetric measurable sets is the same as for open sets. This is proved in [3], as it is a particular case of their general result, but their proof is not easy to read. We describe it here so that a simpler, explanatory proof be available. The constant for open sets has been obtained by Déchamps-Gondim, Piquard-Lust and Queffélec [7, 8], so that

$$\gamma_2 = \sup_{0\leq x} \frac{2\sin^2 x}{\pi x} = 0.4613\cdots . \tag{3}$$

In all proofs, the same kind of estimates as (2), but with finite sums on a grid of points replacing integrals, plays a central role in the proofs. So it was natural to get interested in best constants on these finite structures. This led us to the same problem, but taken on finite groups, which we describe now.

Let us consider $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$, which identifies with the grid (or subgroup) $\mathbb{G}_q := \{k/q; k = 0, 1, \cdots, q-1\}$ contained in the torus. We do not assume that q is a prime number at this point. We still denote by $e(x) := e^{2\pi i x/q}$ the exponential function adapted to the group $\mathbb{Z}_q$ and by e_h the function $e(hx)$. Again the set

$$\mathcal{P}_q := \left\{ \sum_{h\in H} e_h \ : \ H \subset \{0, \cdots, q-1\} \right\} \tag{4}$$

is called the set of *idempotents* on $\mathbb{Z}_q$. In this context, the set of idempotents has 2^q elements.

We then adapt the definition of p-concentration to the setting of $\mathbb{Z}_q$.

Definition 4. *Let $p > 0$. We say that there is uniform (in q) p-concentration for $\mathbb{Z}_q$ if there exists a constant $\gamma > 0$ so that for each prime number q one can find an idempotent $f \in \mathcal{P}_q$ with*

$$2|f(1)|^p \geq \gamma \sum_{k=0}^{q-1} |f(k)|^p. \tag{5}$$

Moreover, writing $\gamma_p^\sharp(q)$ for the maximum of all such constants γ, we put

$$\gamma_p^\sharp := \liminf_{q\to\infty} \gamma_p^\sharp(q).$$

Then $\gamma_p^\sharp$ is called the uniform level of p-concentration.

Here we can formulate a discrete analogue of the problem in [2, 3]. *Does q-uniform concentration fail for $p = 1$?*

The reader may note that in order to define p-concentration in the setting of $\mathbb{Z}_q$, one should also look for f that satisfies (5), but with $f(a)$, for some arbitrary $a \in \mathbb{Z}_q$, in the left hand side. This is easy when q is prime. Indeed, for $a = 0$ the Dirac mass at 0, which is an

idempotent, has the required property with constant 1. Otherwise, if $a \neq 0$ and f satisfies (5), then the function $g(x) := f(a^{-1}x)$ satisfies the same inequality, but with $g(a)$ in the left hand side. Here a^{-1} is the unique inverse for the multiplication in $\mathbb{Z}_q$. Clearly $g(a) = f(1)$, and all other values taken by f are taken by g since multiplication is one-to-one in $\mathbb{Z}_q$ for q prime, so that the right hand side is the same for f and g.

Remark 5. *We can also replace* 1 *by* a *in the left-hand side of* (5) *when* q *is any integer, but* a *and* q *are co-prime.*

As we said, p-concentration on $\mathbb{Z}_q$ plays a role in proofs for p-concentration on the torus. In order to solve the 2-concentration problem on the torus, Déchamps-Gondim, Piquard-Lust and Queffélec [7, 8] have considered the concentration problem on $\mathbb{Z}_q$, proving the precise value that we already mentioned,

$$\gamma_2^\sharp = \sup_{0\leq x} \frac{2\sin^2 x}{\pi x} = 0.4613\cdots . \tag{6}$$

Moreover, they obtained $\gamma_p^\sharp \geq 2(\gamma_2^\sharp/2)^{p/2}$ for all $p > 2$. The last assertion is an easy consequence of the decrease of ℓ^p norms with p, and we have, in general,

$$\gamma_p^\sharp \geq 2(\gamma_{p'}^\sharp/2)^{p/p'} \tag{7}$$

for $p > p'$.

Let us also mention that they considered the same problem for the class of positive definite polynomials, that is

$$\mathcal{P}_q^+ := \left\{ \sum_{h\in H} a_h e_h \ : \ a_h \geq 0, h \in \{0, \cdots, q-1\} \right\}. \tag{8}$$

We say that there is uniform p-concentration on $\mathbb{Z}_q$ for the class of positive definite polynomials if there exists some constant γ such that (5) holds for some $f \in \mathcal{P}_q^+$. We denote by c_p^+ the level of p-concentration for the class of positive definite polynomials, which is defined as the maximum of all admissible constants in (5) (similarly to the class of idempotents).

With these notations, it has been proved in [7] that $c_2^+ = 1/2$. Since the class of positive definite polynomials is stable by taking products, it follows that, for all even integers $2k$,

$$\gamma_{2k}^\sharp \leq c_{2k}^+ \leq 1/2.$$

It is easy to see that there is uniform p-concentration on $\mathbb{Z}_q$ for all $p > 1$, using Dirichlet kernels. This has been used in our paper [5], where the discrete problem under consideration here has been largely studied, at least for p an even integer.

On the other hand, coming back to our main point, i.e. to the case of $p = 1$, and using the recent results of B. Green and S. Konyagin [9], we answer negatively in this case, which gives an affirmative answer to the conjecture of [3] for finite groups $\mathbb{Z}_q$.

All the results on $\mathbb{Z}_q$ summarize in the following theorem, which gives an almost complete answer to the p-concentration problem under consideration, except for the best constants, which are not known for $p \neq 2$.

Theorem 6. *For all $1 < p < \infty$ we have uniform p-concentration on $\mathbb{Z}_q$. We have $\gamma_2^\sharp$ given by* (3)*, then $0.495 < \gamma_4^\sharp \leq 1/2$. For all $p > 2$, we have $\gamma_p^\sharp > 0.483$. On the other hand for $p \leq 1$ we do not have uniform p-concentration.*

Positive results are implicitly contained in [5], where they are used as tools for the problem of concentration on the torus. As far as necessary upper bounds for $\gamma_p^\sharp$ are considered, since the polynomials f with positive coefficients have their maximum at 0, we have the trivial upper bound $\gamma_p^\sharp \leq 2/3$. Moreover, for p an even integer, we have seen that $\gamma_p^\sharp \leq 1/2$. Let us remark that (7) provides an improvement on the bound $2/3$ between two even integers. Indeed, for $p \leq 2k$, we have

$$\gamma_p^\sharp \leq 2^{1-p/k}.$$

In the next two sections, we will consider the case of $\mathbb{Z}_q$, first for $p > 1$, then for $p = 1$. Then, in Section 4, we will come back to the case $p = 2$ on the torus and exploit the proof for giving concentration results by means of the use of the grid $\mathbb{G}_q$. In the last section, we prove Theorem 3.

We tried to keep the notations for the constants the same as in [5], since we refer to the proofs there, and apologize for sometimes these notations seem more complicated than they should be.

2. UNIFORM p-CONCENTRATION

In this section, we will recall the situation on the group $\mathbb{Z}_q$ by transferring the results that have been obtained for the grid

$$\mathbb{G}_q := \{k/q; k = 0, 1, \cdots, q-1\}$$

contained in $\mathbb{T}$. By a slight abuse of notation, let us still denote

$$\mathcal{P}_q := \left\{ \sum_{h \in H} e_h \ : \ H \subset \{0, \cdots, q-1\} \right\} \tag{9}$$

the set of trigonometrical idempotents of degree less than q on $\mathbb{T}$, with e_h denoting the exponential $e_h(x) := e^{2\pi i h x}$ adapted to $\mathbb{T}$. When restricted to $\mathbb{G}_q$ identified with $\frac{1}{q}\mathbb{Z}_q$, it coincides with the corresponding

idempotent (the coefficients are the same, but the exponential is now adapted to $\mathbb{Z}_q$) on $\mathbb{Z}_q$. This is a one-to-one correspondence between idempotents of $\mathbb{Z}_q$ and idempotents of degree less than q, since these last ones are determined by their values on q points, and, in particular, on $\mathbb{G}_q$. We will prefer to deal with ordinary trigonometrical polynomials, and see $\mathbb{Z}_q$ as the grid $\mathbb{G}_q$.

Unless explicitly mentioned, we will only consider Taylor polynomials, that is, trigonometrical polynomials with only non negative frequencies.

We consider the following quantities, written in these new notations, and identify them with the quantities defined for $\mathbb{Z}_q$ in the introduction.

$$\gamma_p^\sharp := \liminf_{q\to\infty} \gamma_p^\sharp(q), \qquad \gamma_p^\sharp(q) := \sup_{R\in\mathcal{P}_q} \frac{2\left|R\left(\frac{1}{q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p}. \tag{10}$$

One can obtain a lower bound of $\gamma_p^\sharp$, with $p>1$, by the only consideration of the Dirichlet kernels

$$D_n(x) := \sum_{\nu=0}^{n-1} e(\nu x) = e^{\pi i(n-1)x}\frac{\sin(\pi n x)}{\sin(\pi x)}. \tag{11}$$

Here the constraint on the degree restricts us to $n<q$. Having n and q tend to infinity with n/q tending to t, we proved in [5] (see Lemma 35) that

Lemma 7. *For $p>1$, we have the inequality*

$$2(\gamma_p^\sharp)^{-1} \le \inf_{0<t<1/2} B(p,t), \tag{12}$$

where, for $\lambda>1$,

$$B(\lambda,t) := \left(\frac{\pi t}{\sin \pi t}\right)^\lambda \left(1+2\sum_{k=1}^\infty \left|\frac{\sin(k\pi t)}{k\pi t}\right|^\lambda\right). \tag{13}$$

It is clear that $B(\lambda,t)$ is bounded for $\lambda>1$, so that $\gamma_p^\sharp>0$ and there is uniform p-concentration: just take as a bound the value for $t=1/4$. Let us try to get more precise estimates. The computation of $\inf_{0<t<1/2} B(\lambda,t)$ can be executed explicitly for $\lambda=2$ and $\lambda=4$. In the first case we recognize in the sum the Fourier coefficients of $\chi_{[-t/2,t/2]}$, whose L^2 norm is $\sqrt{t}$. So (12) leads to the minimization of the function $\frac{2\sin^2 t}{\pi t}$, and to the estimate $\gamma_2^\sharp \ge \sup_{0\le t}\frac{2\sin^2 t}{\pi t} = 0.4613\cdots$. This is the formula given by Déchamps-Gondim, Lust-Piquard and Queffélec in [7]. We refer to them for the necessity of the condition, for which they give a smart proof. For $\lambda=4$, we recognize in the sum of (13) the Fourier coefficients of the convolution product $\chi_{[-t/2,t/2]}*\chi_{[-t/2,t/2]}$,

whose L^2 norm is equal to $(2t^3/3)^{1/2}$. Using Plancherel Formula we obtain that

$$\gamma_4^\sharp \geq \max_{0<t<1/2} \frac{3\left(\sin^4(\pi t)\right)}{\pi^4 t^3} > 0.495. \tag{14}$$

For larger integer values of λ, the computations do not seem to be easily handled. But we can prove that there exists a uniform lower bound for $\gamma_p^\sharp$ when $p \geq 6$. To see this, we need another lemma that can be found in [5]. Let us first give new definitions, relative to positive definite polynomials.

As for idempotents, by the same slight abuse of notation, let us still denote

$$\mathcal{P}_q^+ := \left\{ \sum_{h\in H} a_h e_h \ : \ a_h \geq 0,\, h \in \{0, \cdots, q-1\} \right\}. \tag{15}$$

the set of trigonometrical polynomials with non negative coefficients of degree less than q on $\mathbb{T}$, with e_h denoting the exponential adapted to $\mathbb{T}$. Again, when restricted to $\mathbb{G}_q$, it coincides with the corresponding positive definite polynomial with non negative coefficients on $\mathbb{Z}_q$, and this defines a one-to-one correspondence between positive definite polynomials of $\mathbb{Z}_q$ and positive definite polynomials on $\mathbb{T}$ of degree less than q. The constant c_p^+ can then be defined by

$$c_p^+ := \liminf_{q\to\infty} c_p^+(q), \qquad c_p^+(q) := \sup_{R\in\mathcal{P}_q^+} \frac{2\left|R\left(\frac{1}{q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p}. \tag{16}$$

It is much easier to find positive definite polynomials in $\mathcal{P}_q^+$ than idempotents. In particular, whenever P is in $\mathcal{P}_q$, then, for each positive integer L the polynomial Q, which has degree less than q and has the same values on $\mathbb{G}_q$ as P^L, is in $\mathcal{P}_q^+$. So we can take as well powers of Dirichlet kernels as polynomials R in the right hand side of (16). This leads to the following bounds, using Lemma 7.

$$\begin{aligned} 2(c_p^+)^{-1} &\leq \inf_{L\geq 1} \inf_{0<t<1/2} B(Lp,t) \\ &\leq \inf_{\kappa>0} \limsup_{\lambda\mapsto\infty} B\left(\lambda, \kappa\sqrt{6/\lambda}\right) \\ &\leq 4.13273. \end{aligned} \tag{17}$$

The two last estimates may be found in [5], see (55), and lead to

$$c_p^+ > 0.483. \tag{18}$$

The first one gives a non explicit bound for a fixed p:

$$c_p^+ \geq 2 \sup_{L\geq 1} \sup_{0<t<1/2} B(Lp,t)^{-1}. \tag{19}$$

We prove now that we have the same estimates for $\gamma_p^\sharp$ when $p > 2$.

Theorem 8. *We have $\gamma_p^\sharp > 0.483$ uniformly for all $p > 2$.*

This is a consequence of the following proposition, which is more general than the corresponding results in [5].

Proposition 9. *Let $p > 2$ and $c > 0$, $\varepsilon > 0$. Then there exists $q_0 := q_0(c, \varepsilon)$ such that, if $q > q_0$ and $P := \sum_0^{q-1} a_h e_h$ is a polynomial of degree less than q that satisfies the two conditions*

$$cq \max_h |a_h| \leq \sum |a_h| \leq c^{-1}|P(1/q)|, \tag{20}$$

$$|P(1/q)| \geq c \left(\sum_{k=0}^{q-1} |P(k/q)|^p \right)^{1/p}, \tag{21}$$

then there exists a polynomial Q of degree less than q, whose coefficients are either $a_h/|a_h|$ or 0, such that

$$|Q(1/q)| \geq (1-\varepsilon)|P(1/q)|, \tag{22}$$

$$\left(\sum_{k=0}^{q-1} |Q(k/q) - P(k/q)|^p \right)^{1/p} \leq \varepsilon|P(1/q)|. \tag{23}$$

Observe that, for P positive definite, Q is an idempotent. In this case, the first condition can be reduced to $P(0) \geq cq \max_h |a_h|$. Indeed, the fact that $|P(1/q)| \geq cP(0)$ follows from the second one.

Let us take the proposition for granted, and use it in our context.

Proof of Theorem 8. Let us take for P a positive-definite polynomial of degree less than q for which

$$\frac{2\left|P\left(\frac{1}{q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^p} \geq c_0 > 0.483.$$

We claim that there exists an idempotent Q for which the same ratio is bounded below by $c_0 C(\varepsilon)$, with $C(\varepsilon)$ tending to 1 when ε tends to 0. Indeed, we can apply the proposition as soon as we have proved that P satisfies the condition (20) (uniformly for q large). We have seen that P can be taken as the polynomial of degree less than q, which coincides with D_n^L on the grid $\mathbb{G}_q$, for n chosen in such a way that $n/q \approx t = \kappa\sqrt{6/\lambda}$ is small enough so that we approach the extremum in (17). Next, it is easy to see that $P(0) = n^L$, while $|\hat{P}(k)| \leq Ln^{L-1}$. So we have (20) with a very small constant c, but what is important that it does not depend on q tending to ∞ (for fixed ε). To conclude the proof, we use the fact that, by Minkowski's inequality, and using

the assumption on P, we have

$$\begin{aligned}\left(\sum_{k=0}^{q-1}\left|Q\left(\frac{k}{q}\right)\right|^p\right)^{1/p} &\leq \left(\sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^p\right)^{1/p} + \varepsilon|P(1/q)| \\ &\leq ((2/c_0)^{1/p}+\varepsilon)|P(1/q)| \\ &\leq (1-\varepsilon)((2/c_0)^{1/p}+\varepsilon)|Q(1/q)|.\end{aligned}$$

The constant tends to $(2/c_0)^{1/p}$ when ε tends to 0, which concludes the proof. □

The same method leads to

$$\gamma_p^\sharp \geq 2\sup_{L\geq 1}\sup_{0<t<1/2} B(Lp,t)^{-1}. \tag{24}$$

This finishes the proof of the part of Theorem 6 concerning $p>1$, except for the proof of Proposition 9, which we do now. It relies on the construction of random polynomials, which may have an independent interest.

Proof of Proposition 9. Without loss of generality we may assume that $\max_h |a_h| = 1$. We put $\alpha_k := |a_k|$ and $\sigma := \sum \alpha_k$, so that $0 \leq \alpha_k \leq 1$ and $cq \leq \sigma \leq c^{-1}|P(1/q)|$. We take a sequence of independent random variables $X_0, X_1, \dots, X_{q-1}$ that follow the Bernoulli law with parameters $\alpha_0, \alpha_1, \dots, \alpha_{q-1}$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and set

$$P_\omega := \sum_0^{q-1} b_h X_h(\omega) e_h$$

with $b_h := a_h/|a_h|$ for $a_h \neq 0$, otherwise $b_h = 0$. Then the expectation of P_ω is equal to P. We will prove that $Q = P_\omega$ satisfies (22) and (23) with positive probability. Let us first consider (22), and prove that the converse inequality holds with probability less than $1/3$ for q large enough. Indeed, one has the inclusions

$$\{\omega; |P_\omega(1/q)| \leq (1-\varepsilon)|P(1/q)|\} \subset \{\omega; |P_\omega(1/q) - P(1/q)| > \varepsilon|P(1/q)|\},$$

so that, by Markov inequality, using the fact that the variance of $P_\omega(1/q)$ is $\sum \alpha_k(1-\alpha_k) \leq \sigma$, we have

$$\mathbb{P}\left(\left|\frac{P_\omega(1/q)}{P(1/q)}\right| \leq 1-\varepsilon\right) \leq c^{-2}\varepsilon^{-2}\sigma^{-1}.$$

By (20) we know that this quantity is small for q large.

Next, to show (23), in view of (20) it is sufficient to prove that with probability 2/3,

$$\sum_{k=0}^{q-1} |P_\omega(k/q) - P(k/q)|^p \leq c^p\varepsilon^p\sigma^p.$$

We claim that there exists some uniform constant C_p, for $p > 2$, such that, for each k,

$$\mathbb{E}(|P_\omega(k/q) - P(k/q)|^p) \le C_p \sigma^{p/2}. \tag{25}$$

Let us take this for granted and finish the proof. By simple estimation

$$\mathbb{P}\left(\sum |P_\omega(k/q) - P(k/q)|^p \ge (c\varepsilon\sigma)^p\right) \le c^{-p}\varepsilon^{-p} C_p\, q\, \sigma^{-p/2}.$$

From this we conclude easily, using the fact that $\sigma \ge cq$, so that the right hand side tends to 0 when q tends to infinity. Finally, (25) is a well-known property of independent sums of Bernoulli variables, e.g. in [5] (Lemma 54) a proof of the following lemma can be found.

Lemma 10. *For $p > 2$ there exists some constant C_p with the following property. Let $\alpha_k \in [0, 1]$ and $b_k \in \mathbb{C}$ be arbitrary for $k = 0, 1, \dots, N$. For X_k a sequence of independent Bernoulli random variables with parameter α_k, we have*

$$\mathbb{E}\left(|\sum_{k=0}^{N} b_k(X_k - \alpha_k)|^p\right) \le C_p \cdot \max_{k=1,\dots,N} |b_k|^p \cdot (1 + \sum_{k=0}^{N} \alpha_k)^{p/2}.$$

□

Of course one would like to know whether constants are the same for classes $\mathcal{P}_q$ and $\mathcal{P}_q^+$. We know that it is not the case for $p = 2$ thanks to the work of Déchamps-Gondim, Lust-Piquard and Queffélec, but the last proposition induces to conjecture that they are the same for $p > 2$. Note that Proposition 9 holds when (20) is replaced by the weaker assumption $\sigma \ge \delta(q) q^{2/p} \max |a_h|$, with δ tending to infinity with q.

3. Failure of uniform 1-concentration on $\mathbb{Z}_q$

We prove here the negative result of Theorem 6. It will be more convenient, in this section, to work directly on $\mathbb{Z}_q$, and not on the grid $\mathbb{G}_q$. We now restrict to q prime, which is sufficient to conclude negatively.

Assume that there exists some constant c and some idempotent $f = \sum_{h \in H} e_h$ such that

$$|f(1)| \ge c \sum_{k=0}^{q-1} |f(k)|. \tag{26}$$

We claim that H may be assumed having cardinality $\le q/2$. Indeed, H is certainly not the whole set $\{0, \cdots, q-1\}$, since the corresponding idempotent is q times the Dirac mass at 0. Moreover, the idempotent $\widetilde{f}$, having spectrum ${}^cH$, takes the same absolute values as f outside 0, while its value at 0 is $q - \mathrm{Card}\ H$. So, if $\mathrm{Card}\ H > q/2$, then $\widetilde{f}$ satisfies also (26).

From now on, let $r := \text{Card } H \leq q/2$. We have by assumption (26) $\sum_{k=0}^{q-1} |f(k)| \leq |f(1)|/c \leq f(0)/c = r/c$. So the function

$$g := r^{-1}(f - r\delta_0)$$

is 0 at 0, has ℓ^1 norm bounded by $\frac{1}{c} + 1$, while its Fourier coefficients are equal to $1/r - 1/q$ (r of them), or $-1/q$, since the delta function has all Fourier coefficients equal to $1/q$. But, according to Theorem 1.3 of [9], we should have $q \min_k |\hat{g}(k)|$ tending to 0 when q tends to ∞ (note that the Fourier transform here is replaced by the inverse Fourier transform in [9], which is the reason for multiplication by q compared to the statement given there). This gives a contradiction, and allows to conclude that there is no uniform 1-concentration. This finishes the proof.

We leave the following as an open question.

Problem 11. *In line with Definition 4, for given fixed q denote $\gamma_1^\sharp(q) := \max_{f\in\mathcal{P}_q} 2|f(1)|/\sum_{k=0}^{q-1}|f(k)|$. Determine $\beta := \liminf_{q\to\infty} \log(1/\gamma_1^\sharp(q))/\log\log q$.*

Using the full strength of the result of [9], the constant c in the proof of Theorem 6 may be chosen uniformly bounded from below in q by $\log^{-\alpha} q$, with α less than $1/3$ (that is, the proof by contradiction shows that $c > \log^{-\alpha} q$ is not possible, hence $\beta \geq 1/3$). On the other hand the Dirichlet kernel exhibits $\gamma_1^\sharp(q) \geq C/\log q$, i.e. $\beta \leq 1$. This leaves open the question if β achieves 1, i.e. $\log(1/\gamma_1^\sharp(q))/\log\log q$ can be taken anything less than 1. The problem is in relation with the Littlewood conjecture on groups $\mathbb{Z}_q$, for which there has been new improvements by Sanders [13].

4. 2-concentration on measurable sets

We prove in this section that $\gamma_2 \geq \gamma_2^\sharp$. The converse inequality follows from the fact that the constant for measurable sets is smaller than the one when restricted to open sets, which is $\gamma_2^\sharp$, whose explicit value is given by (6). In this paragraph we shall basically use the method of Anderson et al. [3]. Our improvements are mainly expository. The method is valid for all $p > 1$, and we will write it in this context, even if better results can be obtained for $p \neq 2$. Indeed, it will be easier, later on, to explain how to improve the method starting from this first one.

So we are going to prove the following proposition.

Proposition 12. *For $p > 1$, we have*

$$\gamma_p \geq \gamma_p^\sharp.$$

Proof. We are given an arbitrary symmetric measurable set, with $|E| > 0$. We want to find some idempotent f that concentrates on E. We will

use a variant of Khintchine's Theorem in Diophantine approximation, which we summarize in the next lemma (Proposition 36 in [5]).

Lemma 13. *Let E be a measurable set of positive measure in $\mathbb{T}$. For all $\theta > 0$, $\eta > 0$ and $q_0 \in \mathbb{N}$, there exists an irreducible fraction a/q such that $q > q_0$ and*

$$\left|\left(\frac{a}{q} - \frac{\theta}{q^2}, \frac{a}{q} + \frac{\theta}{q^2}\right) \cap E\right| \geq (1-\eta)\frac{2\theta}{q^2}. \tag{27}$$

Moreover, given a positive integer ν, it is possible to choose q such that $(\nu, q) = 1$.

The parameter θ will play no role at the moment, so we can set it as 1. It will appear as necessary for generalizations only later. We consider the grid $\mathbb{G}_q := \{k/q; k = 0, 1, \cdots, q-1\}$ contained in the torus, for a and q given by Lemma 13, for given values of η and q_0 to be fixed later on. We assume that q is sufficiently large so that we can find $R \in \mathcal{P}_q$ with the property that

$$2|R(a/q)|^p \geq c \sum_{k=0}^{q-1} |R(k/q)|^p, \tag{28}$$

with $\varepsilon > 0$ chosen arbitrarily small and $c > \gamma_p^\sharp - \varepsilon$. When $a = 1$, the existence of such a P follows from the definition of $\gamma_p^\sharp$. See Remark 5 for the fact that we can replace 1 by a whenever a and q are co-prime. We then claim that the polynomial $Q(t) := R(t)D_n(qt)$, which is an idempotent, is such that

$$\int_E |Q|^p \geq c\kappa(\varepsilon) \int_{\mathbb{T}} |Q|^p,$$

with $\kappa(\varepsilon) < 1$ tending to 1 when ε tends to 0, and parameters η and n are chosen suitably depending on ε.

The idea of the proof goes as follows: since D_n concentrates the L^p norm near 0 (it can be concentrated in any subset F of the interval $\left(-\frac{1}{q}, +\frac{1}{q}\right)$, with $|F| > 2(1-\eta)/q$), then $D_n(qt)$ concentrates equally on the q subsets around the points of the grid $\mathbb{G}_q$. We take F such as qt belongs to F when t belongs to $\left(\frac{a}{q} - \frac{\theta}{q^2}, \frac{a}{q} + \frac{\theta}{q^2}\right) \cap E$. Now multiplication by R will concentrate the integral on the subset around a/q, which we wanted. We need to know that the polynomial R is almost constant on each of these subsets, which is given by Bernstein's Theorem.

Let us now enter into details. We have the following lemma on Dirichlet kernels.

Lemma 14. *Let $p > 1$. For ε given, one can find $\eta > 0$ and $\delta_0 > 0$ such that, for all $0 < \delta < \delta_0$, if F is a measurable subset of $(-\delta, +\delta) \subset \mathbb{T}$*

of measure larger than $2\delta(1-\eta)$, *we can find some suitable* $n \in \mathbb{N}$ *so that*

$$\int_F |D_n|^p \geq (1-\varepsilon) \int_{\mathbb{T}} |D_n|^p.$$

Proof. It is well known that $\int_{\mathbb{T}} |D_n|^p \geq \kappa_p n^{p-1}$ (see [3] for instance for precise estimates). So it is sufficient to prove that we can obtain

$$\int_{^cF} |D_n|^p \leq \varepsilon n^{p-1}.$$

This is a consequence of the fact that

$$\int_{(-\delta,+\delta)\setminus F} |D_n|^p \leq 2n^p \eta\delta,$$

while

$$\int_{\mathbb{T}\setminus(-\delta,+\delta)} |D_n|^p \leq \left(\frac{\pi}{2}\right)^p \int_{|t|>\delta} t^{-p} dt = \kappa'_p \delta^{1-p}.$$

We choose for n the smallest integer larger than $(2\kappa'_p/\varepsilon)^{1/(p-1)}\delta^{-1}$ and η such that $8(2\kappa'_p/\varepsilon)^{1/(p-1)}\eta = \varepsilon$.

We remark that here we did not need the flexibility linked to the parameter δ_0. It is here for further generalizations. □

Next we recall classical Bernstein and Marcinkiewicz-Zygmund type inequalities, in the forms tailored to our needs and proved in [5], Lemma 41. Recall that here polynomials are Taylor polynomials, that is, trigonometrical polynomials with only non negative frequencies, which is the case for the polynomial R.

Lemma 15. *For* $1 < p < \infty$ *there exists a constant* K_p *such that, for* P *a polynomial of degree less than* q *and for* $|t| < 1/2$, *we have the two inequalities*

$$\sum_{k=0}^{q-1} |P(t+k/q)|^p \leq K_p \sum_{k=0}^{q-1} |P(k/q)|^p, \tag{29}$$

$$\sum_{k=0}^{q-1} \left||P(t+k/q)|^p - |P(k/q)|^p\right| \leq K_p |qt| \sum_{k=0}^{q-1} |P(k/q)|^p. \tag{30}$$

For our polynomial R, this gives the inequality

$$\left||R(t)|^p - |R(a/q)|^p\right| \leq 2c^{-1} K_p qt |R(a/q)|^p, \tag{31}$$

This implies that, for $|t - \frac{a}{q}| < \frac{\theta}{q^2}$ with q large enough,

$$|R(t)|^p \geq (1-\varepsilon)|R(a/q)|^p. \tag{32}$$

We have also, for $|t| < \frac{\theta}{q^2}$, that

$$\sum_{k=0}^{q-1} |R(t+k/q)|^p \leq \sum_{k=0}^{q-1} |R(k/q)|^p + 2K_p \frac{\theta}{q} c^{-1} |R(a/q)|^p$$

which leads to the inequality, valid for $|t| < \frac{\theta}{q^2}$ for q large enough,

$$\sum_{k=0}^{q-1} |R(t+k/q)|^p \leq 2c^{-1}(1+\varepsilon)|R(a/q)|^p. \tag{33}$$

Let us finally remark that (29) leads to the following, valid for all t.

$$\sum_{k=0}^{q-1} |R(t+k/q)|^p \leq 2c^{-1} K_p |R(a/q)|. \tag{34}$$

We can now proceed to the proof of the required inequality for R. We have fixed ε and chosen q_0 large enough so that estimates (32) and (33) hold (recall that for the moment $\theta = 1$). Then we use Lemma 13, which fixes some a/q, and find D_n, which is assumed to be adapted to $\delta := \frac{\theta}{q}$. We denote $\tau^p := \int_{\mathbb{T}} |D_n|^p$ and $I := \left(\frac{a}{q} - \frac{\theta}{q^2}, \frac{a}{q} + \frac{\theta}{q^2}\right)$.

$$\begin{aligned}
\frac{1}{2}\int_E |Q|^p \geq \int_{I\cap E} |R|^p |D_n|^p &\geq (1-\varepsilon)|R(a/q)|^p \int_{I\cap E} |D_n(qt)|^p dt \\
&\geq \frac{1}{q}(1-\varepsilon)|R(a/q)|^p \int_{F\cap(-\delta,+\delta)} |D_n|^p \\
&\geq \frac{(1-\varepsilon)^2 \tau^p}{q} |R(a/q)|^p. \qquad (35)
\end{aligned}$$

Here F is the pre-image by $t \mapsto qt$ of $I \cap E$, which has measure at least $2(1-\eta)\delta$, and so concentrates the integral of $|D_n|^p$.

Let us now look for a bound of the whole integral. We write

$$\int_{\mathbb{T}} |Q|^p = \int_{-1/q}^{1/q} \left(\sum_k |R(t+\frac{k}{q})|^p \right) |D_n(qt)|^p dt$$

and cut the integral into two parts, depending on the fact that $|t| \leq \frac{\theta}{q^2}$ or not. For the first part we use (33), for the second one (34). We recall that the integral of D_n outside the interval $(-\theta/q, \theta/q)$ is bounded by $\varepsilon\tau^p$. Finally

$$\begin{aligned}
\int_{\mathbb{T}} |Q|^p &\leq 2c^{-1} \frac{1+\varepsilon}{q} |R(a/q)|^p \int_{\mathbb{T}} |D_n|^p + 2c^{-1} K_p \cdot \frac{\varepsilon}{q} |R(a/q)|^p \int_{\mathbb{T}} |D_n|^p \\
&\leq 2c^{-1} \frac{(1+C\varepsilon)\tau^p}{q} |R(a/q)|^p.
\end{aligned}$$

We conclude by comparison with (35). $\square$

As said above, we have obtained optimal results for $p = 2$. At this point, we can see how results can be improved for $p \neq 2$. The main point is the possibility to replace the Dirichlet kernel D_n by an idempotent T, which satisfies nearly the same properties as the Dirichlet kernel that are summarized in Lemma 14, but has the additional property to have arbitrarily large gaps. More precisely, we say that T *has gaps larger than* N if $|k - k'| \leq N$ implies that one of the two Fourier coefficients $\hat{T}(k)$ and $\hat{T}(k')$ is zero. We state the existence of such idempotents T as a lemma, and refer to [5] for their construction.

Lemma 16. *Let $p > 0$ different from 2. Then for $\varepsilon > 0$ there exists $\delta_0 > 0$ and $\eta > 0$ such that, for all $\delta < \delta_0$ and $N \in \mathbb{N}$, if E is a measurable set that satisfies, for $\alpha = 0$, the assumption $|E \cap [\alpha - \delta, \alpha + \delta]| > 2(1 - \eta)\delta$, then there exists an idempotent T with gaps larger than N such that*

$$\int_{E\cap[\alpha-\delta,\alpha+\delta]} |T|^p > (1 - \varepsilon) \int_0^1 |T|^p.$$

Moreover, if p is not an even integer, this is also valid for $\alpha = 1/2$.

For the moment we use this lemma with $\alpha = 0$. We are no more restricted to consider polynomials of degree less than q in order that $R(t)T(qt)$ be an idempotent. It is sufficient that the degree of R be less than Nq, and, since N is arbitrary, this gives essentially no constraint. The fact that R has degree less than q was also used for (32) and (33). It is where the flexibility given by the parameter θ can be used: if R has degree less than q^2, then roughly speaking we can also use Bernstein Inequality, but θ/q has to be replaced by θ in (31). This is of no inconvenience, since θ can be chosen arbitrarily small.

At this point, we could proceed with a polynomial of degree less than q^2 for (32), but certainly not for Lemma 15, since such a polynomial can be identically 0 on the grid $\mathbb{G}_q$. To develop such inequalities for polynomials S of degree larger than q, we will restrict to those that can be written as $S(t) := R(t)R((q + 1)t)$, with R an idempotent that satisfies (28), but for $2p$ instead of p (so that the condition on p is now $p > 1/2$). The important point is that S is also an idempotent, and so is ST if T has sufficiently large gaps. Also $|S(k/q)|^p = |R(k/q)|^{2p}$ at each point of the grid, and in particular at a/q. Moreover, it is easy to see that, for θ small enough, one still has the inequalities (32), (33) and (34) with $2p$ in place of p, both for the polynomials $R(t)$ and $R((q + 1)t)$ (for this last one we have to choose θ small enough, as we mentioned earlier.) The fact that (32), (33) and (34) are valid for S follows from Cauchy-Schwarz Inequality. The rest of the proof goes the same way as the previous one and leads to the following, for which we leave details to the reader.

Proposition 17. *One has p-concentration for $p > 1/2$, and, for $p \neq 2$, one has the inequality $\gamma_p \geq \gamma^\sharp_{2p}$. In particular $\gamma_1 \geq \gamma^\sharp_2$.*

We could as well have taken $S = R_1 R_2$ and used Hölder's Inequality, taking R_1 approaching the maximum concentration on the grid for the exponent r and R_2 approaching the maximum concentration on the grid for the exponent s, with $\frac{p}{r} + \frac{p}{s} = 1$. This leads to the following generalization of the last proposition.

Proposition 18. *One has p-concentration for $p > 1/2$, and, for $p \neq 2$, one has the inequality $\gamma_p \geq \left(\gamma^\sharp_r\right)^{p/r} \left(\gamma^\sharp_s\right)^{p/s}$ for all $r > p$ and $s > p$ such that $\frac{p}{r} + \frac{p}{s} = 1$.*

Before concluding this section, let us make a last observation. Once we use an idempotent T with arbitrarily large gaps, it is not difficult to build idempotents with arbitrarily large gaps. It is sufficient to start from the polynomial $R(\nu t)$, with ν arbitrarily large. Recall that when using Lemma 13, we can take q such that $(\nu, q) = 1$. This means that there exists $b \pmod q$ such that $\nu a = b \pmod q$, and we choose R that satisfies (28), but with b/q in place of a/q. The rest of the proof can be adapted. We state it as a proposition.

Proposition 19. *In Proposition 12 and Proposition 18, when $p \neq 2$, we can have arbitrarily large gaps. That is, when $1/2 < p \neq 2$, given a symmetric measurable set E of positive measure, and any constant $c < \gamma^\sharp_p$ (resp. $\left(\gamma^\sharp_r\right)^{p/r} \left(\gamma^\sharp_s\right)^{p/s}$), there exists an idempotent P with arbitrarily large gaps such that*

$$\int_E |P|^p > c \int_{\mathbb{T}} |P|^p.$$

5. Improvement of constants for p not an even integer

We proved in [5] that $\gamma_p = 1$ for $p > 1$ and p not an even integer. Let us give the main lines of the proof, which will be used again for the improvement of the constant when $p = 1$. As we shall see, it has been slightly simplified compared to the proof in [5]. The main ingredient is the fact that there are idempotents that concentrate as the Dirichlet kernels, but with arbitrarily large gaps, and at $1/2$ instead of 0. We have already stated this in Lemma 16.

If we take such a peaking function T, then $T(qx)$ concentrates around the points of the translated grid

$$\mathbb{G}^\star_q := \frac{1}{2q} + \mathbb{G}_q = \left\{ \frac{2k+1}{2q} \quad ; \quad k = 0, \cdots, q-1 \right\}. \tag{36}$$

We have considerably gained with this new grid compared to $\mathbb{G}_q$ because 0 – where, by positive definiteness, we always must have a maximal value of any idempotent – does not belong to the grid any more,

and thus we will even be able to find idempotents P such that the maximal value of $|P|$ (over the grid) will be attained at the points $\pm 1/2q$, moreover, the sum of the values $|P|^p$ on $\mathbb{G}_q^\star$ is just slightly larger than $2|P(1/2q)|^p$.

Let us interpret the new constants that we will introduce in terms of another concentration problem on a finite group. More precisely, we view $\mathbb{G}_q^\star$ as $\mathbb{G}_{2q} \setminus \mathbb{G}_q$, and identify $\mathbb{G}_{2q}$ with $\mathbb{Z}_{2q}$, while $\mathbb{G}_q^*$ identifies with a coset. Recall that the idempotents on $\mathbb{Z}_{2q}$ are identified with polynomials in $\mathcal{P}_{2q}$. We are interested in relative concentration inside the coset, and give the following definition.

Definition 20. *We define*

$$\Gamma_p^\star := \sup_{K<\infty} \liminf_{q\to\infty} \Gamma_p^\star(q, K), \tag{37}$$

where $\Gamma_p^\star(q,K)$ is the maximum of all constants γ for which there exists $R \in \mathcal{P}_{2q}$ satisfying

$$2\left|R\left(\frac{1}{2q}\right)\right|^p \geq \gamma \sum_{k=0}^{q-1} \left|R\left(\frac{2k+1}{2q}\right)\right|^p \tag{38}$$

$$2\left|R\left(\frac{1}{2q}\right)\right|^p \geq \gamma K^{-1} \sum_{k=0}^{q-1} \left|R\left(\frac{k}{q}\right)\right|^p. \tag{39}$$

In other words, $\Gamma_p^\star$ is positive when there is uniform concentration at $1/2q$, (which is the case for $p > 1$), but the grids $\mathbb{G}_q$ and $\mathbb{G}_q^\star$ do not play the same role; the constant $\Gamma_p^\star$ is only the relative concentration on $\mathbb{G}_q^\star$, which we try to maximize.

Remark 21. *We can also replace* 1 *by* $2a+1$ *in the left-hand side of* (38) *when* q *is any integer, but* $2a+1$ *and* $2q$ *co-primes.*

This is the equivalent of Remark 5. Multiplication by b, such that $b(2a+1) \equiv 1$ modulo $2q$, will send 1 to $2a+1$ and define a bijection on $\mathbb{G}_q^\star$ (resp. $\mathbb{G}_q$).

Lower bounds for $\Gamma_p^\star$ are given in the lemma below, which is a slight modification of Lemma 34 in [5].

Lemma 22. *For $p > 1$, we have the inequality*

$$\frac{1}{\Gamma_p^\star} \leq \inf_{0<t<1/2} A(p,t), \tag{40}$$

where, for $\lambda > 1$,

$$A(\lambda, t) := \frac{1}{(\sin(\pi t))^\lambda} \sum_{k=0}^{\infty} \left|\frac{\sin((2k+1)\pi t)}{2k+1}\right|^\lambda. \tag{41}$$

The inequality is obtained by taking Dirichlet kernels D_n, with $n/2q$ tending to t, a point that will be used later on. Observe that $A(\lambda, t)$ tends to ∞ when t tends to 0, so that the infimum is obtained away from 0. The uniformity in the second inequality (39) is given by a bound of (a small modification of) $B(\lambda, t)$ defined in (13), for which we have the inequality

$$B(\lambda, t) \leq \left(\frac{\pi}{2}\right)^{\lambda} + 2\left(\sum_k k^{-\lambda}\right) t^{-\lambda}. \tag{42}$$

Observe that (for fixed t) $A(\lambda, t)$, and hence also $\inf_{0<t<1/2} A(\lambda, t)$ are decreasing functions of λ. In [5] recognizing the Fourier coefficients (at k and $-k$) of the function $\frac{\pi}{2}\left(\chi_{[-t/2,t/2]}(x) - \chi_{[-t/2,t/2]}(x-1/2)\right)$ we used Plancherel Formula to calculate

$$A(2, t) = \frac{\pi^2 t}{4\sin^2(\pi t)}. \tag{43}$$

Substituting $x = \pi t$ and recalling (3) we find that

$$\Gamma_2^\star \geq 2\gamma_2 \approx 0.9226.$$

Moreover, it is easy to see that $\inf_{0<t<1/2} A(\lambda, t)$ is left continuous in λ at 2, so that

$$\liminf_{p\to 2-0} \Gamma_p^\star \geq 2\gamma_2. \tag{44}$$

Our main estimate for $\Gamma_p^\star$ is the following.

Proposition 23. *For $p > 2$ we have $\Gamma_p^\star = 1$.*

We postpone the proof of this proposition and show how to use it. We need an adaptation of the Khintchine 's type theorem that we used in the last section. The next lemma uses the inhomogeneous extension of Khintchine's Diophantine approximation theorem, first proved by Szüsz [15] and later generalized by Schmidt [14]. This is Proposition 37 of [5].

Lemma 24. *Let E be a measurable set of positive measure in $\mathbb{T}$. For all $\theta > 0$, $\eta > 0$ and $q_0 \in \mathbb{N}$, there exists an irreducible fraction $(2k+1)/(2q)$ such that $q > q_0$ and*

$$\left|\left[\frac{2k+1}{2q} - \frac{\theta}{q^2}, \frac{2k+1}{2q} + \frac{\theta}{q^2}\right] \cap E\right| \geq (1-\eta)\frac{2\theta}{q^2}. \tag{45}$$

Moreover, given a positive integer ν, it is possible to choose q such that $(\nu, q) = 1$.

Our main result is the following.

Theorem 25. *For p not an even integer, one has the inequalities $\gamma_p \geq \Gamma_p^\star$ and $\gamma_p \geq (\Gamma_r^\star)^{p/r} (\Gamma_s^\star)^{p/s}$ for all $r > p$ and $s > p$ such that $\frac{p}{r} + \frac{p}{s} = 1$. Moreover, given a symmetric measurable set E of positive measure, and any constant $c < \Gamma_p^\star$ (resp. $(\Gamma_r^\star)^{p/r} (\Gamma_s^\star)^{p/s}$), there exists an idempotent P with arbitrarily large gaps such that*

$$\int_E |P|^p > c \int_{\mathbb{T}} |P|^p.$$

Proof. We shall first prove the inequality $\gamma_p \geq \Gamma_p^\star$. We will then show how to modify the proof for the other statements.

We are given a symmetric measurable set E. We consider the grid $\mathbb{G}_q^\star = \mathbb{G}_{2q} \setminus \mathbb{G}_q$ contained in the torus, with a and q given by Lemma 24. At this point we have already fixed some $\varepsilon > 0$. The values of q_0, η and θ are also fixed, but we will say how to choose them later on. We assume that q is sufficiently large so that we can find $R \in \mathcal{P}_{2q}$ with the property that

$$2|R(\frac{1}{2q} + \frac{a}{q})|^p \geq c \sum_{k=0}^{q-1} |R(\frac{1}{2q} + \frac{k}{q})|^p, \tag{46}$$

with $c > (1-\varepsilon)\Gamma_p^\star$. Moreover we can assume that

$$\sum_{k=0}^{q-1} |R(\frac{k}{q})|^p \leq 2Kc^{-1} |R(\frac{1}{2q} + \frac{a}{q})|^p \tag{47}$$

for some uniform constant K. The existence of such an R is given by Definition 20 and by the remark just after. Once chosen R, we choose a peaking function T at $1/2$ for the value ε. We assume now that η has been chosen sufficiently small for the existence of such a function T, built for $\delta := \theta/q^2$, which is possible if $\theta q_0^{-2} \leq \delta_0$.

We choose the idempotent $Q(t) := R(t)T(qt)$ (indeed it is an idempotent if T has sufficiently large gaps) and fix $I := \left(\frac{2a+1}{2q} - \frac{\theta}{q^2}, \frac{2a+1}{2q} + \frac{\theta}{q^2}\right)$. We also put $\tau^p := \int_{\mathbb{T}} |T|^p$. From this point on, the proof follows the same lines as the proof of Proposition 12. We have the inequality

$$\begin{aligned}
\frac{1}{2} \int_E |Q|^p \geq \int_{I \cap E} |R|^p |T|^p &\geq (1-\varepsilon) |R((2a+1)/(2q))|^p \int_{I \cap E} |T(qt)|^p dt \\
&\geq \frac{1}{q} (1-\varepsilon) |R((2a+1)/(2q))|^p \int_{F \cap (-\delta, +\delta)} |T|^p \\
&\geq \frac{(1-\varepsilon)^2 \tau^p}{q} |R((2a+1)/(2q))|^p.
\end{aligned}$$

We have used that the pre-image F of $I \cap E$ by $t \mapsto qt$ has measure at least $2(1-\eta)\delta$, and concentrates the integral of $|T|^p$ at $1/2$. We have

also used the inequality,

$$|R(t)|^p \geq (1-\varepsilon)|R(\frac{2a+1}{2q})|^p, \tag{48}$$

valid for $|t-\frac{2a+1}{2q}| < \frac{\theta}{q}$ with θ small enough. This is an easy consequence of Lemma 16 for polynomials of degree $2q$, since the sum of values of $|R|^p$ on the whole grid $\mathbb{G}_{2q}$ is bounded by $2c^{-1}(K+1)$ times its value at $(2a+1)/(2q)$. Just take θ small enough (we fix θ in such a way that this is valid).

Before going on, let us remark that the other two basic inequalities can be deduced from Lemma 16. First, for $|t-\frac{2a+1}{2q}| < \frac{\theta}{q}$ with θ small enough, we have also

$$\sum_{k=0}^{q-1} |R(t+\frac{k}{q})|^p \leq 2c^{-1}(1+\varepsilon)|R(\frac{2a+1}{2q})|^p. \tag{49}$$

Finally, for all t, we have, for some constant κ,

$$\sum_{k=0}^{q-1} |R(t+\frac{k}{q})|^p \leq \kappa|R(\frac{2a+1}{2q})|^p. \tag{50}$$

Here we can take $\kappa := 2c^{-1}K_p(K+1)$. Next we look for a bound of the whole integral

$$\int_{\mathbb{T}} |Q|^p = \int_0^{1/q} \left(\sum_k |R(t+\frac{k}{q})|^p\right) |T(qt)|^p dt$$

and cut the integral into two parts, depending on the fact that $|t-\frac{1}{2q}| \leq \frac{\theta}{q}$ or not. For the first part we use (49), for the second one (50). We recall that the integral of T outside the interval $(\frac{1}{2}-\frac{\theta}{q}, \frac{1}{2}+\frac{\theta}{q})$ is bounded by $\varepsilon\tau^p$.

$$\begin{aligned}\int_{\mathbb{T}} |Q|^p &\leq 2c^{-1}\,\frac{1+\varepsilon}{q}|R(\frac{2a+1}{2q})|^p\ \tau^p + \kappa\frac{\varepsilon}{q}|R(\frac{2a+1}{2q})|^p\tau^p \\ &\leq 2c^{-1}\,\frac{(1+C\varepsilon)\tau^p}{q}|R(\frac{2a+1}{2q})|^p.\end{aligned}$$

We conclude by comparison with the integral on E. This allows to conclude for the first case, $\gamma_p \geq \Gamma_p^\star$.

Let us now indicate the necessary modification for finding $\gamma_p \geq (\Gamma_r^\star)^{p/r}\,(\Gamma_s^\star)^{p/s}$. In the following we denote $r_1 := r$ and $r_2 := s$: the index j will always cover the two values $j=1$ and $j=2$. Instead of starting from one polynomial, we start from two polynomials R_1 and

R_2 in $\mathcal{P}_{2q}$, which satisfy the following inequalities, for $j = 1, 2$.

$$2|R_j(\frac{2a+1}{2q})|^{r_j} \geq c_j \sum_{k=0}^{q-1} |R_j(\frac{2a+1}{2q})|^{r_j}, \tag{51}$$

with $c_j > (1-\varepsilon)\Gamma^\star_{r_j}$. Moreover we assume that

$$\sum_{k=0}^{q-1} |R_j(\frac{k}{q})|^{r_j} \leq 2Kc^{-1}|R_j(\frac{2a+1}{2q})|^{r_j} \tag{52}$$

for some uniform constant K. We then put $R(t) := R_1(t)R_2((2q+1)t)$. We remark that, on $\mathbb{G}_{2q}$, the values of R coincide with the values of the product R_1R_2. We will prove that we still have inequalities (48) and (49) for $|t - \frac{2a+1}{2q}| < \frac{\theta}{q^2}$, and (50) for all t. Let us first prove that (50) holds for some constant κ. Indeed, by Hölder Inequality with conjugate exponents r_1/p and r_2/p and periodicity of R_2, we have

$$\sum_{k=0}^{q-1} |R(t+\frac{k}{q})|^p \leq \left(\sum_{k=0}^{q-1} |R_1(t+\frac{k}{q})|^{r_1}\right)^{\frac{p}{r_1}} \times \left(\sum_{k=0}^{q-1} |R_2((2q+1)t+\frac{k}{q})|^{r_2}\right)^{\frac{p}{r_2}}.$$

Both factors are bounded, up to a constant, respectively by $|R_1(\frac{2a+1}{2q})|^p$ and $|R_2(\frac{2a+1}{2q})|^p$, which allows to conclude.

In view of (48) and (49), we remark that, when t differs from $\frac{2a+1}{2q}$ by less than $\frac{\theta}{q^2}$, then $(2q+1)t$ differs from $\frac{2a+1}{2q}$ (modulo 1) by less than $\frac{3\theta}{q}$. So we still have, for $|t - \frac{2a+1}{2q}| < \frac{\theta}{q^2}$ with θ small enough,

$$|R(t)|^p \geq (1-\varepsilon)|R(\frac{2a+1}{2q})|^p. \tag{53}$$

For Inequality (49), we first use Hölder Inequality with conjugate exponents r_1/p and r_2/p as before, then the same kind of estimate for each factor.

From this point, the proof is the same.

It remains to indicate how to modify the proof to get peaking idempotents with arbitrarily large gaps. So we fix ν as a large odd integer, and we will prove that we can replace the polynomial R used above by some

$$S(x) := R_1(\nu x)R_2((2q+1)\nu x),$$

which has gaps larger than ν. Recall first that we can take arbitrarily large q satisfying $(\nu, q) = 1$, and get an idempotent by multiplication by $T(qx)$ for T having sufficiently large gaps. The value taken by the polynomial S at $\frac{2a+1}{2q}$ is the value of R_1R_2 at $\frac{2b+1}{2q}$, with $\nu(2a+1) \equiv 2b+1$ mod $2q$. So we choose R_1 and R_2 as before, but with b in place of a.

From this point the proof is identical, apart from an additional factor ν, which modifies the value of θ. We know that $S(\nu x)$ and $R(x)$take

globally the same values on both grids $\mathbb{G}_q$ and $\mathbb{G}_q^\star$, because in each case we multiply by an odd integer that is coprime with $2q$. $\square$

Now Theorem 3 is an easy consequence of Proposition 23 and Theorem 25: take $r < 2$ and $s > 2$, so that $\gamma_1 \geq 1 \cdot (\Gamma_r^\star)^{1/r}$, and take the limit of Γ_r for $r \to 2 - 0$ using (44).

Proof of Proposition 23. The proof is in the same spirit as the proof of the inequality $\gamma_p^\sharp > 0.483$. Let us first fix $c < 1$ and prove that we can find a positive definite polynomial of degree less than $2q$ such that

$$2|P(\frac{1}{2q} + \frac{a}{q})|^p \geq c \sum_{k=0}^{q-1} |P(\frac{1}{2q} + \frac{k}{q})|^p,$$

while

$$2|P(\frac{1}{2q} + \frac{a}{q})|^p \geq c \sum_{k=0}^{q-1} |P(\frac{k}{q})|^p.$$

Indeed, it is proved in [5] (and elementary) that $A(Lp, 1/4)$ has limit $1/2$ when L tends to ∞, which means that we can take for P a polynomial that coincides with D_n^L on the grid $\mathbb{G}_{2q}$. We fix L large enough, and choose n to be approximately $q/4$. The second inequality follows from (42).

At this point one can use Proposition 9, with q replaced by $2q$, to find the idempotent Q.

$\square$

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